Beam Loading in Bunch Trains

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Because of beam loading, each bunch in a bunch train will see different RF fields. Thus, each bunch is essentially having its motion governed by a different Hamiltonian. If all bunches are injected at the same reference point in their respective buckets, some of them must be injected unmatched. In particular, the center of the bunch distribution will not be the fixed point in the RF bucket. This leads to the bunch undergoing synchrotron oscillations about the fixed point of its Hamiltonian, and eventually filamenting into a bunch with a larger emittance and a different distribution center than it originally had. In this paper we compute the fixed point, oscillation amplitude, and emittance growth for a bunch in a chain of linacs and arcs. This paper does not treat the problem self-consistently: the preceding bunches are assumed to be fixed in their buckets, and eigenmodes of collective oscillation for the bunch train are not computed.

I. SMOOTHED HAMILTONIAN

Let's begin with the Hamiltonian

$$H = -\frac{1}{2}A_{56}\Delta^2 + \frac{q\bar{v}}{\omega}\sin(\omega\tau + \bar{\phi}) - \frac{qv}{\omega}(\omega\tau\cos\phi + \sin\phi). \tag{1}$$

This is a Hamiltonian which is being accelerated with a gradient $\bar{v}$ where the reference particle arrives at a phase $\bar{\phi}$ behind the crest or the RF. Δ is the difference in energy from that of a reference particle which is accelerated with a gradient v and arriving at a phase $\bar{\phi}$ behind the crest of the RF. τ is the difference in arrival time from the reference particle accelerated with gradient v and phase ϕ. For the purposes of this discussion, let's consider v, $\bar{v}$, ϕ, $\bar{\phi}$ and A_{56} to be constant along the beam line.

In general, the fixed point of this Hamiltonian will not be at $\tau = 0$. We can compute the equations of motion for this Hamiltonian:

$$\frac{d\tau}{ds} = A_{56}\Delta \qquad \frac{d\Delta}{ds} = q\bar{v}\cos(\omega\tau + \bar{\phi}) - qv\cos\phi. \tag{2}$$

The fixed point for this Hamiltonian is $\Delta = 0$ and at τ given by the solution of the equation

$$\bar{v}\cos(\omega\tau + \bar{\phi}) = v\cos\phi. \tag{3}$$

If $\bar{v} > v\cos\phi$, then there is a solution of this equation, which is

$$\omega\tau = \cos^{-1}\left(\frac{v}{\bar{v}}\cos\phi\right) - \bar{\phi}. \tag{4}$$

If $\bar{v} < v\cos\phi$, there is no solution to this equation. Thus, if beam loading reduces the gradient so much that $\bar{v} < v\cos\phi$, the bunch will be lost. The Hamiltonian linearized about this fixed point, if it exists, is

$$-\frac{1}{2}\frac{q}{\omega}\sqrt{\bar{v}^2 - v^2\cos^2\phi^2}\left[\omega\tau - \cos^{-1}\left(\frac{v}{\bar{v}}\cos\phi\right) + \bar{\phi}\right]^2 - \frac{1}{2}A_{56}\Delta^2. \tag{5}$$

Now let's say we're injecting a bunch at $\tau = \tau_0$ and $\Delta = 0$, with the covariance matrix

$$\Sigma = \begin{bmatrix}\sigma_{\tau\tau} & \sigma_{\tau\Delta}\\ \sigma_{\tau\Delta} & \sigma_{\Delta\Delta}\end{bmatrix}. \tag{6}$$

The transformation U is

$$U = \begin{bmatrix}B^{-1/4} & 0\\ 0 & B^{1/4}\end{bmatrix} \qquad B = \frac{q\omega\sqrt{\bar{v}^2 - v^2\cos^2\phi^2}}{A_{56}} \tag{7}$$

The result is that

$$\langle J\rangle = \frac{1}{2}(B^{1/2}\sigma_{\tau\tau} + B^{-1/2}\sigma_{\Delta\Delta}) + \frac{1}{2}\frac{B^{1/2}}{\omega^2}\left[\omega\tau_0 - \cos^{-1}\left(\frac{v}{\bar{v}}\cos\phi\right) + \bar{\phi}\right]^2. \tag{8}$$

This allows the computation of the emittance growth.

Let's take a simple example: say the bunch is injected at $\tau_0 = 0$, and is properly matched to the RF without beam loading (i.e., $\bar{v} = v$ and $\bar{\phi} = \phi$). Then the emittance will blow up to

$$\frac{1}{2}\epsilon_L\left[\frac{(\bar{v}^2 - v^2\cos^2\phi)^{1/4}}{\sqrt{v\sin\phi}} + \frac{\sqrt{v\sin\phi}}{(\bar{v}^2 - v^2\cos^2\phi)^{1/4}}\right] + \frac{1}{2\omega^2}\sqrt{\frac{q\omega\sqrt{\bar{v}^2 - v^2\cos^2\phi}}{A_{56}}}\left[\bar{\phi} - \cos^{-1}\left(\frac{v}{\bar{v}}\cos\phi\right)\right]^2. \tag{9}$$

Assuming that the changes in the RF parameters are small, this becomes

$$\epsilon_L + \frac{\epsilon_L}{8}\left(\frac{\Delta v}{v}\right)^2\csc^4\phi + \frac{1}{2\omega^2}\sqrt{\frac{q\omega v\sin\phi}{A_{56}}}\left[\frac{\Delta(v\cos\phi)}{v\sin\phi}\right]^2. \tag{10}$$

A Beam Loading

If a charge q passes through the linac at a phase ϕ, the change in voltage and phase is given by the relation

$$\Delta(v\cos\phi) = \frac{q\omega r_s}{2Q} \tag{11}$$

and $\Delta(v\cos\phi) = \Delta v/\cos\phi$. First, let's determine under what conditions the beam will still be captured by the RF. The requirement that $\bar{v} > v\cos\phi$ translates into

$$v^2 - \frac{qr_s\omega}{Q}v\cos\phi + \left(\frac{qr_s\omega}{2Q}\right)^2 > v\cos\phi. \tag{12}$$

Solving for v, this means that

$$v > \frac{qr_s\omega}{2Q}\csc^2\phi\left[\cos\phi + \sqrt{\cos(2\phi)}\right]. \tag{13}$$

First of all, this indicates that there is a stable solution when $|\phi| > 45°$.

When Δv is small, the relative error in the emittance

$$\frac{1}{8}\left(\frac{q\omega r_s\cos\phi}{2Qv}\right)^2\csc^4\phi + \frac{1}{2}\left(\frac{qr_s}{2Q\sigma_\tau v\sin\phi}\right)^2. \tag{14}$$

We can similarly compute the maximum average energy that the beam will have, assuming that the beam does not filament. That mean energy is

$$\sqrt{\frac{q\sqrt{\bar{v}^2 - v^2\cos^2\phi}}{\omega A_{56}}}\left|\omega\tau_0 - \cos^{-1}\left(\frac{v}{\bar{v}}\cos\phi\right) + \bar{\phi}\right|. \tag{15}$$

Again, assuming a bunch injected with $\tau_0 = 0$ and small Δv, we get that the maximum energy error is

$$\sqrt{\frac{\omega}{A_{56}qv\sin\phi}}\frac{q^2r_s}{2Q}. \tag{16}$$

This gives an energy error relative to the RMS energy spread of

$$\frac{qr_s}{2Q\sigma_\tau v\sin\phi}. \tag{17}$$

II. EXAMPLES

Let's take an example where we scale Tesla cavities; at 1300 MHz, $r_s/Q = 1$ kΩ/m and $v = 25$ MV/m. We scale r_s/Q with the square root of frequency and v with the square root of frequency. Assume 2×10^{12} muons. Table I shows the results. Note that the figures assume that the central bunch with beam loading has the right energy gain, so effectively only half of the particles contribute to the energy offset and emittance blowup.

APPENDIX A: EMITTANCE COMPUTATION

Begin with the computation of some important quantities. Note that if A is an $n\times n$ matrix, and $\boldsymbol{x}$ is a n-dimensional vector,

$$\frac{1}{(2\pi)^{n/2}}\frac{1}{\sqrt{\det A}}\int e^{-\boldsymbol{x}^TA\boldsymbol{x}/2}d\boldsymbol{x} = 1 \tag{A1}$$

$$\frac{1}{(2\pi)^{n/2}}\frac{1}{\sqrt{\det A}}\int \boldsymbol{x}\boldsymbol{x}^Te^{-\boldsymbol{x}^TA\boldsymbol{x}/2}d\boldsymbol{x} = A^{-1}. \tag{A2}$$

Let's say that the linear transfer map for a beamline is M. Further, let's say that there is a symplectic transformation U such that $M = URU^{-1}$ where R is a rotation matrix. That rotation matrix can be broken into coordinate pairs such that projecting out that pair of coordinates,

$$R_k = P_kRP_k = \begin{bmatrix}\cos\mu_k & \sin\mu_k\\ -\sin\mu_k & \cos\mu_k\end{bmatrix} \tag{A3}$$

Then the action variables can be defined as

$$J_k = \frac{1}{2}\boldsymbol{x}^TU^{-1^T}P_kU^{-1}\boldsymbol{x}. \tag{A4}$$

Computing the expectation value for J_k when the particles are distributed according to a Gaussian distribution with covariance matrix Σ, and offset $\boldsymbol{x}_0$, we find

$$\langle J_k\rangle = \frac{1}{2}\operatorname{Tr}(P_kU^{-1}\Sigma U^{-1^T}P_k) + \frac{1}{2}\boldsymbol{x}_0^TU^{-1^T}P_kU^{-1}\boldsymbol{x}_0 \tag{A5}$$

APPENDIX B: BUCKET AREA

Consider the Hamiltonian

$$H = -\frac{1}{2}A_{56}\Delta^2 + \frac{qv}{\omega}[\sin(\omega\tau+\phi) - \omega\tau\cos\phi - \sin\phi]. \tag{B1}$$

We wish to find the area of the RF bucket. At the bucket separatrix ($\Delta = 0$, $\omega\tau = -2\phi$), the value of the Hamiltonian is

$$2(\phi\cos\phi - \sin\phi) \tag{B2}$$

The value of Δ for the separatrix at any point τ is

$$\sqrt{\frac{2qv}{\omega A_{56}}}\sqrt{\sin(\omega\tau+\phi) + \sin\phi - (2\phi+\omega\tau)\cos\phi} \tag{B3}$$

Thus, the bucket area will be

$$\sqrt{\frac{2qv}{\omega^3A_{56}}}f(\phi), \tag{B4}$$

TABLE I: Neutrino factory example.

$p_{\min}c$	$p_{\min}c$	f	n	ϕ	σ_τ	σ_Δ	ΔE	$\Delta E/\sigma_\Delta$	$\Delta E/E$	$\Delta\epsilon_L/\epsilon_L$
GeV	GeV	MHz		$^\circ$	ps	MeV	MeV	%	%	%
3	12	200	4	26	111	84	2.2	2.6	0.018	0.03
3	12	200	6	29	126	74	1.5	2.1	0.013	0.02
3	12	200	11	35	153	61	0.9	1.4	0.007	0.01
3	12	200	22	43	188	50	0.5	1.0	0.004	0.00
3	12	400	4	32	69	136	6.6	4.9	0.055	0.12
3	12	400	5	34	74	127	5.5	4.3	0.045	0.09
3	12	400	6	36	78	120	4.7	3.9	0.039	0.08
3	12	400	11	43	94	100	2.8	2.8	0.023	0.04
3	12	400	4	40	43	220	20.0	9.1	0.167	0.42
3	12	400	5	42	46	206	16.8	8.2	0.140	0.34
12	50	200	4	16	70	135	8.8	6.5	0.018	0.22
12	50	200	6	19	80	118	5.7	4.8	0.011	0.12
12	50	200	11	23	97	97	3.2	3.3	0.006	0.06
12	50	200	23	29	123	77	1.6	2.1	0.003	0.02
12	50	400	4	20	44	215	25.4	11.8	0.051	0.71
12	50	400	6	23	50	188	17.1	9.1	0.034	0.42
12	50	400	11	28	61	155	9.6	6.2	0.019	0.20
12	50	400	23	35	76	123	5.0	4.1	0.010	0.08
12	50	800	4	26	27	343	72.8	21.2	0.146	2.30
12	50	800	6	29	31	302	50.5	16.7	0.101	1.43
12	50	800	11	35	38	250	28.2	11.5	0.058	0.68
12	50	800	23	43	47	201	15.8	7.8	0.032	0.31

where

$$f(\phi) = \int_{-2\phi}^{\phi_{\max}(\phi)} \sqrt{\sin(x+\phi) + \sin\phi - (2\phi + x)\cos\phi} \tag{B5}$$

and $\phi_{\max}(\phi)$ is the value of x near ϕ where the argument of the square root is zero.

For small ϕ, we can find a series expansion for $\phi_{\max}(\phi)$

$$\phi_{\max} \approx \phi + \frac{\phi^3}{10} + \frac{29}{1400}\phi^5 + \frac{221}{42000}\phi^7 + \frac{346,943}{232,848,000}\phi^9 + O(\phi^{11}) \tag{B6}$$

and for $f(\phi)$:

$$f(\phi) \approx \frac{6\sqrt{2}}{5}\phi^{5/2} - \frac{\sqrt{2}}{50}\phi^{9/2} + \frac{1607\sqrt{2}}{254800}\phi^{13/2} + O(\phi^{17/2}). \tag{B7}$$

The procedure for obtaining these expansions is as follows: first, find an expansion $\phi_{\max}(\phi)$ out to order $2k+1$ in ϕ. Next, expand the integrand divided by $2y+x$ in powers of $[\phi_{\max}(\phi) - x]$ to order $k+1/2$, keeping the coefficient of the lth power out to order $k+1/2-l$ in ϕ. The individual terms can now be integrated, and the result is good to order $k+5/2$ in ϕ.

REFERENCES

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